

Jacobi's derivative formula with modular forms of level five

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Abstract: In this paper, we derive another expressions of Jacobi's derivative formula in terms of modular forms of level five. For this purpose, we use the residue theorem and the Fourier series expansions of $\eta^5(\tau)/\eta(5\tau)$ and $\eta^5(5\tau)/\eta(\tau)$, where $\eta(\tau)$ is the Dedekind eta function.

Key Words: theta function; theta constant; rational characteristics; Hecke group.

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1 Introduction

Throughout this paper, the *upper half plane* \mathbb{H}^2 is defined by $\mathbb{H}^2 = \{\tau \in \mathbb{C} \mid \Im \tau > 0\}$.

The *Dedekind eta function* is defined by $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$, $q = \exp(2\pi i\tau)$, $\tau \in \mathbb{H}^2$.

Our concern is with the famous product-series identities,

$$\frac{\eta^5(\tau)}{\eta(5\tau)} = 1 - 5 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \left(\frac{d}{5} \right) \right) q^n, \quad (1.1)$$

$$\frac{\eta^5(5\tau)}{\eta(\tau)} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{n}{d} \left(\frac{d}{5} \right) \right) q^n, \quad q = \exp(2\pi i\tau), \quad (1.2)$$

where for each $m \in \mathbb{N}$,

$$\left(\frac{m}{5} \right) = \begin{cases} 1 & \text{if } m \equiv \pm 1 \pmod{5}, \\ -1 & \text{if } m \equiv \pm 2 \pmod{5}, \\ 0 & \text{if } m \equiv 0 \pmod{5}. \end{cases}$$

Ramanujan's famous identity,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^5}{(1 - q^n)^6},$$

is proved by the identity (1.2), where for each $n \in \mathbb{N}$, $p(n)$ is the partition number of n . For the proof, see Bailey [1, 2].

The aim of this paper is to obtain another expressions of *Jacobi's derivative formula* by means of the identities (1.1) and (1.2). Jacobi's formula is given by

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\pi \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (1.3)$$

Moreover, we derive modular equations of level 5 and the formulas of Wronskian of modular forms of level 5. Our main theorem is as follows:

Theorem 1.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \theta^{11} \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^{11} \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta^{10} \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} - 11 \theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} - \theta^{10} \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}, \quad (1.4)$$

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \zeta_5^4 \theta^{11} \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \theta^{11} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta^{10} \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} - 11 \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} - \theta^{10} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}, \quad (1.5)$$

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \theta^{11} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^{11} \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - 11 \zeta_5^4 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} - \zeta_5^3 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}, \quad (1.6)$$

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \theta^{11} \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} \theta^{11} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} + 11 \zeta_5 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} - \zeta_5^2 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}, \quad (1.7)$$

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \zeta_5^3 \theta^{11} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^{11} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - 11 \zeta_5^3 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} - \zeta_5 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}, \quad (1.8)$$

and

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 = \frac{2^4 \pi^4 \zeta_5^3 \theta^{11} \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} \theta^{11} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} - 11 \zeta_5 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} - \zeta_5^2 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix}}, \quad (1.9)$$

where $\zeta_5 = \exp(2\pi i/5)$.

This paper is organised as follows. In Section 2, we review the basic properties of the theta functions.

In Section 3, we prove the identities (1.1) and (1.2). In Section 4, we recall the derivative formulas of level five. In [4] and [5], we expressed $\theta' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}$ by theta constants with rational characteristics for $\epsilon = 1, \epsilon' \in \mathbb{Q}$.

In Sections 5 and 6, we prove the formulas (1.4) and (1.5). In Section 7, we prove the formulas (1.6), (1.7), (1.8), and (1.9). For the proof, we note the residue theorem.

Theorem 1.2. (The residue theorem) *The sum of all the residues of an elliptic function in the fundamental parallelogram is zero.*

2 The properties of the theta functions

2.1 Notations

Following Farkas and Kra [3], we first introduce the *theta function with characteristics*, which is defined by

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta) := \sum_{n \in \mathbb{Z}} \exp \left(2\pi i \left[\frac{1}{2} \left(n + \frac{\epsilon}{2} \right)^2 \tau + \left(n + \frac{\epsilon}{2} \right) \left(\zeta + \frac{\epsilon'}{2} \right) \right] \right),$$

where $\epsilon, \epsilon' \in \mathbb{R}$, $\zeta \in \mathbb{C}$, and $\tau \in \mathbb{H}^2$. The *theta constants* are given by

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0, \tau).$$

Let us denote the derivative coefficients of the theta functions by

$$\theta' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \frac{\partial}{\partial \zeta} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) \Big|_{\zeta=0}, \quad \theta'' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \frac{\partial^2}{\partial \zeta^2} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) \Big|_{\zeta=0}, \quad \theta''' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} := \frac{\partial^3}{\partial \zeta^3} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) \Big|_{\zeta=0}.$$

2.2 Basic properties

We first note that for $m, n \in \mathbb{Z}$,

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta + n + m\tau, \tau) = \exp(2\pi i) \left[\frac{n\epsilon - m\epsilon'}{2} - m\zeta - \frac{m^2\tau}{2} \right] \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau), \quad (2.1)$$

and

$$\theta \begin{bmatrix} \epsilon + 2m \\ \epsilon' + 2n \end{bmatrix} (\zeta, \tau) = \exp(\pi i \epsilon n) \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau). \quad (2.2)$$

Furthermore, it is easy to see that

$$\theta \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (\zeta, \tau) = \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\zeta, \tau) \text{ and } \theta' \begin{bmatrix} -\epsilon \\ -\epsilon' \end{bmatrix} (\zeta, \tau) = -\theta' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\zeta, \tau).$$

We note that $\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau)$ has only one zero in the fundamental parallelogram, which is given by

$$\zeta = \frac{1 - \epsilon}{2} \tau + \frac{1 - \epsilon'}{2}.$$

2.3 Jacobi's triple product identity

All the theta functions have infinite product expansions, which are given by

$$\begin{aligned} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) &= \exp \left(\frac{\pi i \epsilon \epsilon'}{2} \right) x^{\frac{\epsilon^2}{4}} z^{\frac{\epsilon}{2}} \\ &\times \prod_{n=1}^{\infty} (1 - x^{2n}) (1 + e^{\pi i \epsilon'} x^{2n-1+\epsilon} z) (1 + e^{-\pi i \epsilon'} x^{2n-1-\epsilon} / z), \end{aligned} \quad (2.3)$$

where $x = \exp(\pi i \tau)$ and $z = \exp(2\pi i \zeta)$. Therefore, it follows from Jacobi's derivative formula (1.3) that

$$\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0, \tau) = -2\pi q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)^3 = -2\pi \eta^3(\tau), \quad q = \exp(2\pi i \tau).$$

2.4 The heat equation

The theta function satisfies the following heat equation:

$$\frac{\partial^2}{\partial \zeta^2} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau) = 4\pi i \frac{\partial}{\partial \tau} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta, \tau). \quad (2.4)$$

3 On $\eta^5(\tau)/\eta(5\tau)$ and $\eta^5(5\tau)/\eta(\tau)$

3.1 Some theta functional formulas

Proposition 3.1. *For every $(z, \tau) \in \mathbb{C} \times \mathbb{H}^2$, we have*

$$\begin{aligned} & \theta^2 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z) \theta \begin{bmatrix} 1 \\ \frac{9}{5} \end{bmatrix} (z) - \theta^2 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z) \theta \begin{bmatrix} 1 \\ \frac{7}{5} \end{bmatrix} (z) \\ & + \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z) = 0, \end{aligned} \quad (3.1)$$

$$\begin{aligned} & -\zeta_5^2 \theta^2 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (z) \theta \begin{bmatrix} \frac{9}{5} \\ 1 \end{bmatrix} (z) + \zeta_5^3 \theta^2 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (z) \theta \begin{bmatrix} \frac{7}{5} \\ 1 \end{bmatrix} (z) \\ & + \theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z) = 0. \end{aligned} \quad (3.2)$$

Proof. We prove equation (3.1). Equation (3.2) can be proved in the same way. We first note that $\dim \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 2$, and

$$\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{9}{5} \end{bmatrix} (z, \tau), \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{7}{5} \end{bmatrix} (z, \tau), \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) \in \mathcal{F}_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, there exist some complex numbers, x_1, x_2 and x_3 not all zero such that

$$x_1 \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{9}{5} \end{bmatrix} (z, \tau) + x_2 \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z, \tau) \theta \begin{bmatrix} 1 \\ \frac{7}{5} \end{bmatrix} (z, \tau) + x_3 \theta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z, \tau) = 0.$$

Note that in the fundamental parallelogram, the zero of $\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z)$, $\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z)$, or $\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$ is $z = 2/5, 1/5$ or 0 . Substituting $z = 2/5, 1/5$, and 0 , we have

$$\begin{aligned} & x_2 \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} + x_3 \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} = 0, \\ & -x_1 \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} + x_3 \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} = 0, \\ & -x_1 \theta^2 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} - x_2 \theta^2 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} = 0. \end{aligned}$$

Solving this system of equations, we have

$$(x_1, x_2, x_3) = \alpha \left(\theta^2 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}, -\theta^2 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}, \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{5}{3} \end{bmatrix} \right) \text{ for some } \alpha \in \mathbb{C} \setminus \{0\},$$

which proves the proposition. \square

Lemma 3.2. *For every $(z, \tau) \in \mathbb{C} \times \mathbb{H}^2$, we have*

$$\left\{ \frac{\theta' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z)} \right\}^2 = \frac{\theta'' \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z)}{\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z)} - \frac{d^2}{dz^2} \log \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z).$$

Proof. The lemma can be proved by direct calculation. \square

3.2 Proof the identities (1.1) and (1.2)

Proof. By equation (3.1), we derive equation (1.1). Equation (1.2) can be proved in the same way.

Comparing the coefficients of the term z^2 in equation (3.1), we have

$$\frac{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = \frac{d^2}{dz^2} \log \theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z) \Big|_{z=0} - \frac{d^2}{dz^2} \log \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z) \Big|_{z=0}.$$

Therefore, equation (1.1) can be obtained by Jacobi's triple product identity (2.3). \square

4 Derivative formulas of level five

From Matsuda [5], recall the following derivative formulas:

Theorem 4.1. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{\theta' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} = \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - 3\zeta_5^4 \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} \right)}{10\theta^3 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}, \quad \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} = \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\left(3\theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} + \zeta_5^4 \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} \right)}{10\theta^3 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}.$$

5 Proof of the formula (1.4)

5.1 Proof of the formula (1.4)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (z) \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (z) \theta \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

In the fundamental parallelogram, the pole of $\varphi(z)$ or $\psi(z)$ is $z = 0$, which implies that $\text{Res}(\varphi(z), 0) = \text{Res}(\psi(z), 0) = 0$. Therefore, it follows that

$$2 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} + \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 4 \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} \right\}^2 = 0, \quad (5.1)$$

and

$$\frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} + 2 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} - 4 \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} \right\}^2 = 0. \quad (5.2)$$

By the drivative formulas of Theorem 4.6, we have

$$\frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = \frac{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{50\theta^6 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^6 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} \left\{ -4\theta^{10} \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} + 44\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} + 4\theta^{10} \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \right\}. \quad (5.3)$$

The heat equation (2.4) and Jacobi's triple product identity (2.3) yields

$$\begin{aligned} \frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} &= 4\pi i \frac{d}{d\tau} \log \frac{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = -8\sqrt{5}\pi^2 \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{n}{d} \left(\frac{d}{5} \right) \right) q^n, \quad q = \exp(2\pi i\tau), \\ &= -8\sqrt{5}\pi^2 \frac{\eta^5(5\tau)}{\eta(\tau)} = -\frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}. \end{aligned}$$

Therefore, it follows that

$$-\frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2} = \frac{-2\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{25\theta^6 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^6 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} \left\{ \theta^{10} \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} - 11\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} - \theta^{10} \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} \right\},$$

which proves the formula (1.4). \square

Corollary 5.1. *For every $\tau \in \mathbb{H}^2$, set $q = \exp(2\pi i\tau)$. Then, we have*

$$\begin{aligned} \frac{22\sqrt{5}}{50} \frac{\eta^5(\tau)}{\eta(5\tau)} + 5\sqrt{5} \frac{\eta^5(5\tau)}{\eta(\tau)} &= \frac{25 + 11\sqrt{5}}{50} \left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1+\sqrt{5}}{2}q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2 \\ &\quad - \frac{25 - 11\sqrt{5}}{50} \left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1-\sqrt{5}}{2}q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2. \end{aligned}$$

Proof. The corollary follows from Jacobi's triple product identity (2.3) and equation (5.3). \square

5.2 Theorem of Farkas and Kra

Theorem 5.2. (Farkas and Kra [3, pp. 318]) *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{d}{d\tau} \log \left(\frac{\eta(5\tau)}{\eta(\tau)} \right) + \frac{1}{2\pi i \cdot 3} \left[\left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (0, \tau)} \right\}^2 + \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (0, \tau)} \right\}^2 \right] = 0.$$

Proof. Summing both sides of equations (5.1) and (5.2) yields

$$3 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} + 3 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} - 2 \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} \right\}^2 + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} \right\}^2 = 0.$$

The heat equation (2.4) implies that

$$4\pi i \frac{d}{d\tau} \log \frac{\theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2} + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} \right\}^2 + 2 \left\{ \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} \right\}^2 = 0.$$

Jacobi's triple product identity (2.3) yields

$$\frac{\theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^3 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2} = \frac{(\sqrt{5})^3 \eta^3(5\tau)}{4\pi^2 \eta^3(\tau)},$$

which proves the theorem. \square

Corollary 5.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$1 + 6 \sum_{n=1}^{\infty} (\sigma(n) - 5\sigma(n/5)) q^n = \frac{25 + 11\sqrt{5}}{50} \left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1+\sqrt{5}}{2} q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2 \\ + \frac{25 - 11\sqrt{5}}{50} \left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1-\sqrt{5}}{2} q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2.$$

Proof. The corollary follows from Jacobi's triple product identity (2.3) and the derivative formulas of Theorem 4.6. \square

5.3 Some product-series identities

Theorem 5.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1+\sqrt{5}}{2} q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2 = 1 + \frac{25 - 11\sqrt{5}}{4} \sum_{n=1}^{\infty} \left\{ 30 \sum_{\substack{d|n \\ 5 \nmid d}} d + \sqrt{5} \sum_{d|n} \left(\frac{d}{5}\right) \left(25 \frac{n}{d} - 11d\right) \right\} q^n,$$

and

$$\left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^5 \left(1 + \frac{1-\sqrt{5}}{2} q^n + q^{2n}\right)^5}{(1 - q^{5n})^3} \right\}^2 = 1 + \frac{25 + 11\sqrt{5}}{4} \sum_{n=1}^{\infty} \left\{ 30 \sum_{\substack{d|n \\ 5 \nmid d}} d - \sqrt{5} \sum_{d|n} \left(\frac{d}{5}\right) \left(25 \frac{n}{d} - 11d\right) \right\} q^n.$$

Proof. The theorem follows from Corollaries 5.1 and 5.3. □

5.4 The modular equation of level 5.

Theorem 5.5. *For every $\tau \in \mathbb{H}^2$, set*

$$(X, Y, Z) = \left(\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (0, \tau), \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (0, \tau), \frac{\eta^5(\tau)}{\eta(5\tau)} \right).$$

Then, (X, Y, Z) satisfies the following relations:

$$5^5 X^9 Y^9 - Z^{10} (X^2 - 11XY - Y^2)^5 = 0. \quad (5.4)$$

Proof. The formula (1.4) implies that

$$Z^2 (X^2 - 11XY - Y^2) = 5\theta^9 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta^9 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}.$$

Equation (5.4) follows from raising to the fifth power both sides of this equation. □

5.5 The formula of Wronskian

Theorem 5.6. *For every $\tau \in \mathbb{H}^2$, set*

$$(X, Y, Z) = \left(\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (0, \tau), \theta^5 \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} (0, \tau), \frac{\eta^5(\tau)}{\eta(5\tau)} \right).$$

Then, (X, Y, Z) satisfies the following relations:

$$\frac{dX}{d\tau} Y - X \frac{dY}{d\tau} = \frac{2\pi i}{(\sqrt{5})^3} Z (X^2 - 11XY - Y^2). \quad (5.5)$$

Proof. Note that

$$5 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix}} = 4\pi i \frac{d}{d\tau} \log X, \quad 5 \frac{\theta'' \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = 4\pi i \frac{d}{d\tau} \log Y,$$

and

$$\frac{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{\theta \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix}} = \frac{4\pi^2 \eta^5(\tau)}{\sqrt{5} \eta(5\tau)} = \frac{4\pi^2}{\sqrt{5}} Z.$$

Equation (5.5) follows from (5.3). □

Theorem 5.7. For every $\tau \in \mathbb{H}^2$, set

$$(X, Y) = \left(\theta^5 \begin{bmatrix} 1 \\ \frac{1}{5} \end{bmatrix} (0, \tau), \theta^5 \begin{bmatrix} \frac{1}{3} \\ \frac{5}{5} \end{bmatrix} (0, \tau) \right), W(X, Y) = \left| \frac{X}{\frac{dX}{d\tau}} \quad \frac{Y}{\frac{dY}{d\tau}} \right|.$$

Then, the Wronskian of (X, Y) is given by

$$W(X, Y)^{10} = \left(\frac{2\pi i}{5} \right)^{10} X^9 Y^9 (X^2 - 11XY - Y^2)^5. \quad (5.6)$$

Proof. The theorem can be proved by eliminating Z from Theorems 5.5 and 5.6. \square

6 Proof of the formula (1.5)

6.1 Proof of the formula (1.5)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (z) \theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (z) \theta \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

In the fundamental parallelogram, the pole of $\varphi(z)$ or $\psi(z)$ is $z = 0$, which implies that $\text{Res}(\varphi(z), 0) = \text{Res}(\psi(z), 0) = 0$. Therefore, it follows that

$$2 \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} + \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 4 \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} \right\}^2 = 0, \quad (6.1)$$

and

$$\frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} + 2 \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} - 4 \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} \right\}^2 = 0. \quad (6.2)$$

By the drivative formulas of Theorem 4.3, we have

$$\frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} = \frac{\zeta_5 \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{50\theta^6 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^6 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} \left\{ 4\theta^{10} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} + 44\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} - 4\theta^{10} \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \right\}. \quad (6.3)$$

The heat equation (2.4) and Jacobi's triple product identity (2.3) yields

$$\begin{aligned} \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} &= 4\pi i \frac{d}{d\tau} \log \frac{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} = -\frac{8\pi^2}{25} \left(1 - 5 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \left(\frac{d}{5} \right) \right) y^n \right), \quad y = \exp \left(\frac{2\pi i \tau}{5} \right), \\ &= -\frac{8\pi^2}{25} \frac{\eta^5(\tau/5)}{\eta(\tau)} = -\frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}. \end{aligned}$$

Therefore, it follows that

$$-\frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2} = \frac{2\zeta_5 \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{25\theta^6 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^6 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} \left\{ \theta^{10} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} + 11\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} - \theta^{10} \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} \right\},$$

which proves the formula (1.5). \square

Corollary 6.1. *For every $\tau \in \mathbb{H}^2$, set $q = \exp(2\pi i \tau)$. Then, we have*

$$\left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{5n-1})^5 (1 - q^{5n-4})^5} \right\}^2 - \left\{ q \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{5n-2})^5 (1 - q^{5n-3})^5} \right\}^2 = 11 \frac{\eta^5(5\tau)}{\eta(\tau)} + \frac{\eta^5(\tau)}{\eta(5\tau)}.$$

Proof. Equation (6.3) and Jacobi's triple product identity yield

$$\left\{ \prod_{n=1}^{\infty} \frac{(1 - y^n)^2}{(1 - y^{5n-1})^5 (1 - y^{5n-4})^5} \right\}^2 - \left\{ y \prod_{n=1}^{\infty} \frac{(1 - y^n)^2}{(1 - y^{5n-2})^5 (1 - y^{5n-3})^5} \right\}^2 = 11 \frac{\eta^5(\tau)}{\eta(\tau/5)} + \frac{\eta^5(\tau/5)}{\eta(\tau)},$$

where $y = \exp(2\pi i \tau/5)$. The corollary can be obtained by changing $\tau \rightarrow 5\tau$. \square

6.2 Theorem of Farkas and Kra type

Theorem 6.2. *For every $\tau \in \mathbb{H}^2$, we have*

$$\frac{d}{d\tau} \log \left(\frac{\eta(\tau/5)}{\eta(\tau)} \right) + \frac{1}{2\pi i \cdot 3} \left[\left\{ \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, \tau)} \right\}^2 + \left\{ \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, \tau)}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, \tau)} \right\}^2 \right] = 0.$$

Proof. Summing both sides of equations (6.1) and (6.2) yields

$$3 \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} + 3 \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} - 2 \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} \right\}^2 + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} \right\}^2 = 0.$$

The heat equation (2.4) implies that

$$4\pi i \frac{d}{d\tau} \log \frac{\theta^3 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^3 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} \right\}^2 + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} \right\}^2 = 0.$$

Jacobi's triple product identity (2.3) yields

$$\frac{\theta^3 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^3 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}^2} = \frac{\zeta_5^3}{4\pi^2} \frac{\eta^3(\tau/5)}{\eta^3(\tau)},$$

which proves the theorem. □

Corollary 6.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\left\{ \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{5n-1})^5 (1-q^{5n-4})^5} \right\}^2 + \left\{ q \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^{5n-2})^5 (1-q^{5n-3})^5} \right\}^2 = 1 + 6 \sum_{n=1}^{\infty} (\sigma(n) - 5\sigma(n/5)) q^n.$$

Proof. The corollary follows from Theorems 4.3 and 6.2. □

6.3 Some product-series identities

Theorem 6.4. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

$$\left\{ \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{5n-1})^5 (1 - q^{5n-4})^5} \right\}^2 = 1 + \sum_{n=1}^{\infty} \left\{ 3 \sum_{\substack{d|n \\ 5 \nmid d}} d + \frac{1}{2} \sum_{d|n} \left(\frac{d}{5} \right) \left(11 \frac{n}{d} - 5d \right) \right\} q^n,$$

and

$$\left\{ q \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{5n-2})^5 (1 - q^{5n-3})^5} \right\}^2 = \sum_{n=1}^{\infty} \left\{ 3 \sum_{\substack{d|n \\ 5 \nmid d}} d - \frac{1}{2} \sum_{d|n} \left(\frac{d}{5} \right) \left(11 \frac{n}{d} - 5d \right) \right\} q^n.$$

Proof. The theorem follows from Corollaries 6.1 and 6.3. □

6.4 The modular equation of level 5

Theorem 6.5. *For every $\tau \in \mathbb{H}^2$, set*

$$(X, Y, Z) = \left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, 5\tau), \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, 5\tau), \frac{\eta^5(5\tau)}{\eta(\tau)} \right).$$

Then, (X, Y, Z) satisfies the following relations:

$$X^9 Y^9 - Z^{10} (X^2 + 11XY - Y^2)^5 = 0. \tag{6.4}$$

Proof. Set

$$(\tilde{X}, \tilde{Y}, \tilde{Z}) = \left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, \tau), \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, \tau), \frac{\eta^5(\tau)}{\eta(\tau/5)} \right).$$

The formula (1.4) implies that

$$\tilde{Z}^2 (\tilde{Y}^2 - 11\tilde{X}\tilde{Y} - \tilde{X}^2) = \zeta_5 \theta^9 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta^9 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}.$$

Equation (6.4) follows from raising to the fifth power both sides of this equation and changing $\tau \rightarrow 5\tau$. □

6.5 The formula of Wronskian

Theorem 6.6. *For every $\tau \in \mathbb{H}^2$, set*

$$(X, Y, Z) = \left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, 5\tau), \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, 5\tau), \frac{\eta^5(5\tau)}{\eta(\tau)} \right).$$

Then, (X, Y, Z) satisfies the following relations:

$$\frac{dX}{d\tau}Y - X\frac{dY}{d\tau} = 2\pi i Z(X^2 + 11XY - Y^2). \quad (6.5)$$

Proof. Set

$$(\tilde{X}, \tilde{Y}, \tilde{Z}) = \left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, \tau), \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, \tau), \frac{\eta^5(\tau)}{\eta(\tau/5)} \right).$$

Note that

$$5 \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}} = 4\pi i \frac{d}{d\tau} \log \tilde{X}, \quad 5 \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} = 4\pi i \frac{d}{d\tau} \log \tilde{Y},$$

and

$$\frac{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{\theta \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix}} = \frac{4\pi^2}{\zeta_5} \frac{\eta^5(\tau)}{\eta(\tau/5)} = \frac{4\pi^2}{\zeta_5} \tilde{Z}.$$

Equation (6.3) implies that

$$\frac{d\tilde{X}}{d\tau}\tilde{Y} - \tilde{X}\frac{d\tilde{Y}}{d\tau} = \frac{2\pi i}{5}\tilde{Z}(\tilde{X}^2 + 11\tilde{X}\tilde{Y} - \tilde{Y}^2).$$

Equation (6.5) can be obtained by changing $\tau \rightarrow 5\tau$. □

Theorem 6.7. *For every $\tau \in \mathbb{H}^2$, set*

$$(X, Y) = \left(\theta^5 \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} (0, 5\tau), \theta^5 \begin{bmatrix} \frac{3}{5} \\ 1 \end{bmatrix} (0, 5\tau) \right), \quad W(X, Y) = \begin{vmatrix} X & Y \\ \frac{dX}{d\tau} & \frac{dY}{d\tau} \end{vmatrix}.$$

Then, the Wronskian is given by

$$W(X, Y)^{10} = (2\pi i)^{10} X^9 Y^9 (X^2 + 11XY - Y^2)^5. \quad (6.6)$$

Proof. The theorem can be proved by eliminating Z from Theorems 6.5 and 6.6. □

7 Proof of the formulas (1.6), (1.7), (1.8) and (1.9)

7.1 Proof of the formula (1.6)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} (z) \theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} (z) \theta \begin{bmatrix} -\frac{1}{5} \\ -\frac{1}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

In the fundamental parallelogram, the pole of $\varphi(z)$ or $\psi(z)$ is $z = 0$, which implies that $\text{Res}(\varphi(z), 0) = \text{Res}(\psi(z), 0) = 0$. Therefore, it follows that

$$2 \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} + \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} + 4 \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} \right\}^2 = 0,$$

and

$$\frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} + 2 \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta''' \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}} - 4 \frac{\theta' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} \cdot \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} + 2 \left\{ \frac{\theta' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} \right\}^2 = 0.$$

By the drivative formulas of Theorem 4.1, we have

$$\frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} = \frac{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{50\theta^6 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^6 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} \left\{ 4\theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - 44\zeta_5^4 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} - 4\zeta_5^3 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} \right\}.$$

The heat equation (2.4) and Jacobi's triple product identity (2.3) yields

$$\begin{aligned} \frac{\theta'' \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} - \frac{\theta'' \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} &= 4\pi i \frac{d}{d\tau} \log \frac{\theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}} = -\frac{8\pi^2}{25} \left(1 - 5 \sum_{n=1}^{\infty} \left(\sum_{d|n} d \left(\frac{d}{5} \right) \right) \tilde{y}^n \right) \\ &= -\frac{8\pi^2}{25} \frac{\eta^5((\tau+1)/5)}{\eta(\tau+1)} = \frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}, \end{aligned}$$

where $\tilde{y} = \exp \left\{ \frac{2\pi i(\tau + 1)}{5} \right\}$. Therefore, it follows that

$$\frac{32\pi^4}{25} \frac{\theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}}{\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2} = \frac{2\theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2}{25\theta^6 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^6 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix}} \left\{ \theta^{10} \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} - 11\zeta_5^4 \theta^5 \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} \theta^5 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} - \zeta_5^3 \theta^{10} \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} \right\},$$

which proves the formula (1.6). \square

7.2 Proof of the formula (1.7)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} \frac{1}{5} \\ \frac{3}{5} \end{bmatrix} (z) \theta \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} \frac{3}{5} \\ \frac{3}{5} \end{bmatrix} (z) \theta \begin{bmatrix} -\frac{1}{5} \\ -\frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

The formula (1.7) can be proved in the same way as the formula (1.6) \square

7.3 Proof of the formula (1.8)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \end{bmatrix} (z) \theta \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} (z) \theta \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

The formula (1.8) can be proved in the same way as the formula (1.6) \square

7.4 Proof of the formula (1.9)

Proof. Consider the following elliptic functions:

$$\varphi(z) = \frac{\theta^2 \begin{bmatrix} \frac{1}{5} \\ \frac{9}{5} \end{bmatrix} (z) \theta \begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}, \quad \psi(z) = \frac{\theta^2 \begin{bmatrix} \frac{3}{5} \\ \frac{7}{5} \end{bmatrix} (z) \theta \begin{bmatrix} -\frac{1}{5} \\ \frac{1}{5} \end{bmatrix} (z)}{\theta^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)}.$$

The formula (1.9) can be proved in the same way as the formula (1.6) \square

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